On characterization of the extension property

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The geometric characterization of the extension property for Cantor-type sets, found in [3], is related to the rate of growth of the values of the discrete logarithmic energies of compact sets that locally form the set.

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1. Extension Problem

Let K be a compact set in $\mathbb{R}^d$. Then $\mathcal{E}(K)$ is the space of Whitney jets on K, that is the space of traces on K of C^∞ functions. Topology in the space $\mathcal{E}(K)$ can be given by the system of seminorms

$$||\, f\, ||_q = inf \ |\, F\, |_q, \quad q \in \mathbb{N},$$

where the infimum is taken for all possible extensions of f to F and $|\, F\, |_q$ denotes the $q-$th norm of F in $C^\infty(\mathbb{R}^d)$.

The Extension Problem is to characterize when there exists a linear continuous extension operator $L : \mathcal{E}(K) \longrightarrow C^\infty(\mathbb{R}^d)$. We say that the compact set K has the extension property if there exists a such operator.

Tidten in [6] applied Vogt's condition for a splitting of exact sequences of Fréchet spaces and gave a topological characterization of the extension property: a compact set K has the extension property if and only if the space $\mathcal{E}(K)$ has a dominating norm (see for instance [2] for the definition of a dominating norm and for a recent account of the theory).

Nevertheless, the problem of a geometric characterization of the extension property that goes back to the work [5] of Mityagin, is still open even for the one-dimensional case, in spite of the presence of numerous particular results. Here we consider the geometric characterization of the extension property for Cantor-type sets found in [3].

2. Cantor-type sets

Given l_1 with $0 < l_1 < 1/2$ and a sequence $(\alpha_s)_{s=2}^{\infty}$ with $\alpha_s > 1$ let us define the sequence $(l_s)_{s=0}^{\infty}$ in the following way: $l_0 = 1, l_1, l_2 = l_1^{\alpha_2}, \cdots, l_s = l_1^{\alpha_2\alpha_3\cdots\alpha_s}, \cdots$. Then by $K^{(\alpha_n)}$ we denote the symmetric Cantor-type set $\bigcap_{s=0}^{\infty}\bigcup_{j=1}^{2^s} I_{j,s}$, where $|I_{j,s}| = l_s$ for all j. Here the closed intervals $I_{j,s}$ we call *basic* intervals. Let x be an endpoint of some basic interval. Then there exists the minimal number q (the *type* of x) such that x is the endpoint of some $I_{j,m}$ for every $m \geq q$.

As in [3] we suppose that $\alpha_s \geq 1 + \varepsilon_0, s \geq s_0$ for some positive ε_0 and $l_s \geq 4\,l_{s+1}$ for all s. Let $h_s = l_s - 2l_{s+1}$ be the gap between two adjacent intervals.

We follow the notations used in [3]: $\pi_{n,0} = 1$ and for $n \geq 1, s \geq 1$ let

$$\pi_{n,s} = 2^{-s}\alpha_{n+1}\alpha_{n+2}\cdots\alpha_{n+s}, \qquad \sigma_{n,s} = \sum_{k=0}^{s}\pi_{n,k}.$$

Theorem [3]. The following are equivalent:
(i) The set $K^{(\alpha_n)}$ has the extension property.
(ii) $\forall M > 0 \quad \exists\, s_M : \quad l_{n+s}^M > l_n^{2^s} l_{n+1}^{2^{s-1}} \cdots l_{n+s}, \quad \forall n \ \ \forall s \geq s_M.$
(iii) $\sigma_{n,s+1} \,/\, \sigma_{n,s} \rightrightarrows 1, \quad$ as $\quad s \to \infty$ uniformly with respect to n.

We see that the condition (ii) is purely geometrical, whereas the condition (iii) is related to the theory of logarithmic potential. In what follows *log* denotes the natural logarithm.

3. Discrete logarithmic energies

Let K be a compact set in $\mathbb{C}$ and for given N points $z_1, \cdots, z_N \subset K$ let $\mu_N = \mu_N(z_1, \cdots, z_N)$ denote the discrete measure that associates the mass $1/N$ to any point z_k, $1 \leq k \leq N$. The logarithmic potential of measure μ_N is given by

$$U^{\mu_N}(z) = \frac{1}{N}\sum_{k=1}^{N} log\frac{1}{|z - z_k|}.$$

Any discrete measure has infinite energy, but if we use the truncated kernels (see e.g.[1]), then we can define the corresponding logarithmic energy as in [4]:

$$I(\mu_N) = \frac{1}{N^2}\sum_{k,j=1,k\neq j}^{N} log\frac{1}{|z_j - z_k|}.$$

Clearly,

$$I(\mu_N) = \frac{-2}{N^2} \ \ log|\, V(z_1, \cdots, z_N)|,$$

where for the corresponding Vandermonde determinant we have

$$V(z_1, \cdots, z_N) = \prod_{1 \le j < k \le N} (z_k - z_j).$$

Points $(z_k)_{k=1}^N \subset K$ that maximize the determinant $V(z_1, \cdots, z_N)$ (or minimize the corresponding discrete logarithmic energy) are known as Fekete points of order N for K.

Given compact set $K^{(\alpha_n)}$ let us fix $n \in \mathbb{N}$ and $N = 2^{s+1}$ for some $s \in \mathbb{N}_0 := \{0, 1, \cdots\}$.

Let $(z_k)_{k=1}^N$ consist of all endpoints of the type $\le s+n$ on the first basic interval $[0, l_n]$ ordered increasingly, that is $z_1 = 0, z_2 = l_{s+n}, z_3 = l_{s+n-1} - l_{s+n}, \cdots, z_N = l_n$.

Lemma 3.1.

$$e^{-2^{2s+3}\, l_n^{\varepsilon_0}} \ \ l_n^{4^s\, \sigma_{n,\,s}} < |\, V(z_1, \cdots, z_N)| < \, l_n^{4^s\, \sigma_{n,\,s}}.$$

Proof. Temporarily we denote $l_{n+s}\, l_{n+s-1}^2 \cdots l_n^{2^s}$ by λ. We see at once that $\prod_{k=2}^N |\, z_k - z_1\,| < \lambda$. The product $\prod_{k=3}^N |\, z_k - z_2\,|$ with the upper bound $l_{n+s-1}^2 \cdots l_n^{2^s}$ we join with $|\, z_N - z_{N-1}\,| = l_{n+s}$. We continue in this fashion to join the product $\prod_{k=m+1}^N |\, z_k - z_m\,|$ with the product $\prod_{k=N-m+2}^N |\, z_k - z_{N-m+1}\,|$ for $m = 3, \cdots, 2^s$. For each m we can estimate from above the joint product through λ. Since there are 2^s pairs of products, we get the bound

$$|\, V(z_1, \cdots, z_N)| < \lambda^{2^s}.$$

By definition, λ has the form $l_n^{\varkappa}$, where $\varkappa = \alpha_{n+1}\, \alpha_{n+2} \cdots \alpha_{n+s} + 2\alpha_{n+1} \cdots \alpha_{n+s-1} + \cdots + 2^{s-1}\, \alpha_{n+1} + 2^s = 2^s\, [1 + 2^{-1}\alpha_{n+1} + \cdots + 2^{-s}\alpha_{n+1}\, \alpha_{n+2} \cdots \alpha_{n+s}] = 2^s\, \sigma_{n,\,s}$, which gives the desired upper bound of the lemma.

To estimate $|\, V(z_1, \cdots, z_N)|$ from below we replace all l_k in λ by h_k for $k = n, n+1, \cdots, n+s-1$, since the distance between any two points z_i, z_j that belong to the same basic interval of the length l_k, but do not belong to an interval of the length l_{k+1}, is not smaller than h_k.

Therefore,

$$|\, V(z_1, \cdots, z_N)| > (l_{n+s}\, h_{n+s-1}^2 \cdots h_n^{2^s})^{2^s} = \lambda^{2^s} \cdot a$$

with $log\,a = 2^s\,(2\,log\,\frac{h_{n+s-1}}{l_{n+s-1}} + \cdots + 2^s\,log\,\frac{h_n}{l_n})$. By condition, $\frac{h_k}{l_k} = 1 - 2\frac{l_{k+1}}{l_k} > \frac{1}{2}$. From this $log\,\frac{h_k}{l_k} > -4\,\frac{l_{k+1}}{l_k} \geq -4\,l_k^{\varepsilon_0}$ and $log\,a > -2^s\sum_{k=n}^{n+s-1} 2^{n+s-k+2}\,l_k^{\varepsilon_0} > -2^{2s+3}\,l_n^{\varepsilon_0}$, which completes the proof. $\square$

Corollary.

$$\frac{1}{2}\,\sigma_{n,\,s}\,log\,\frac{1}{l_n} < I(\mu_N) < \frac{1}{2}\,\sigma_{n,\,s}\,log\,\frac{1}{l_n} + 4\,l_n^{\varepsilon_0}. \tag{1}$$

Theorem 3.1. *If the set $K^{(\alpha_n)}$ has the extension property then*

$$I_n(\mu_{2^{s+1}})/I_n(\mu_{2^s}) \to 1, \quad as \quad s \to \infty, n \to \infty.$$

Here $I_n(\mu_{2^{s+1}})$ stands for the discrete logarithmic energy defined by all end-points of the type $\leq s + n$ on any basic interval of the length l_n.

Proof. Write $\gamma_n = 8\,l_n^{\varepsilon_0}\ log^{-1}\,l_n^{-1}$. From (1) we have

$$\frac{I_n(\mu_{2^{s+1}})}{I_n(\mu_{2^s})} < \frac{\sigma_{n,\,s} + \gamma_n}{\sigma_{n,\,s-1}} < \frac{\sigma_{n,\,s}}{\sigma_{n,\,s-1}} + \gamma_n,$$

since $\sigma_{n,\,s-1} > 1$. Now the result follows on the condition (iii) and decrease of the sequence γ_n. $\square$

One can conjecture that the existence of a linear continuous extension operator for the space $\mathcal{E}(K)$ (at least for Cantor-type sets) is characterized by a regularity of growth of the minimal discrete logarithmic energies corresponding to compact sets that locally form the set K. The points $(z_k)_{k=1}^N$ considered in Lemma give rather rough approximation of the minimal energy for the set $K^{(\alpha_s)} \cap [0, l_n]$. The exact position of the Fekete points is not known even for rarefied Cantor-type sets.

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